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Thermal shock waves induced by a moving crack—a heat flux formulation

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Abstract—The thermal shock phenomena induced by a rapidly propagating crack tip in a solid medium was studied recently via the temperature formulation (*ASME J. Heat Transfer* **112**, 21–27 (1990)). In order to further confirm the unique features obtained in the previous research for the thermal waves emanating from the crack tip, the present work employs the flux formulation and further examines the same physical phenomena from a different viewpoint. The thermal shock discontinuities result from the energy accumulation in a preferential direction as the speed of the propagating crack tip exceeds the heat propagation speed in the solid. By employing the flux formulation in the thermal wave model, the present work investigates the thermal shock formation, the evolutions of the heat affected zone and the thermally undisturbed zone, and the transition of the r-dependency of the heat flux vector in transition of the thermal Mach number from the subsonic to the supersonic ranges. The analogy between the temperature and the heat flux in the near-tip region is established and the thermal shock phenomena are confirmed from a theoretical point of view.

INTRODUCTION

THE THERMAL wave model describes the process of heat transport by wave phenomena propagating at a finite speed in the solid. The physical essence of this model has been investigated from various points of view. They include, for example, a collision model established on the basis of statistical mechanics [2], identification of the analogy between the random walk and the diffusion processes [3], modification of thermodynamics with fading memory [4], consideration of special relativity for the heat transport process [5, 6], and an interpretation in terms of kinetic theory of molecules [7, 8]. In these works, the limitations of the thermal diffusion model were also examined from different viewpoints. The mathematical structure of the field equations in the hyperbolic theory of heat conduction was also investigated. For a one-dimensional solid carrying the thermal energy with a finite speed, the research developed in this sense includes the propagation of thermal waves in semi-infinite media [9–12], propagation and reflection in a finite medium [13–15], the analytical studies on the wave character under high heat flux [16], the effects of sudden change of thermal properties across dissimilar media in contact [17, 18], and the thermal wave characteristics across a thin surface layer [19]. The wave solutions for the hyperbolic heat conduction equation were also discussed from both analytical [20] and numerical [21, 22] approaches. Generally speaking, significant deviations between the diffusion and the wave models were found in these studies as (i) the transient time is short, (ii) the operational temperature is low, and (iii) the temperature gradient established in the material volume is large. Condition (iii) is a common feature for the problems involving an interface between dissimilar materials, high heat flux, or a thin surface layer.

In a recent work [1], the fundamental characteristics of the thermal wave in the vicinity of a moving crack tip were investigated. The near-tip *temperature* was obtained around the crack tip subjected to a temperature-specified condition at the crack surfaces. The major findings in ref. [1] can be summarized as follows. (1) The thermal field around the moving crack tip can be characterized by a *thermal Mach number* (M) which weighs the ratio of the crack speed (v) to the heat propagation speed (C) in the solid. Mathematically, $M = v/C$. (2) At the *transonic* and in the *supersonic* ranges with $M \geqslant 1$, *thermal shock waves* are present in the physical domain which separate the *heat affected zone* from the *thermally undisturbed zone*. (3) For cases with $M \geqslant 1$, the physical domain of the heat affected zone is $0 \leqslant \theta \leqslant \sin^{-1}(1/M)$, with the angle θ being measured from the crack surface. The thermal shock waves are located at $\theta = \pm\sin^{-1}(1/M)$ which is defined as the *thermal shock angle*. (4) The near-tip temperature has a *discontinuous* but *finite* change across the surface of the thermal shock wave. It jumps from a value of $4(M^2-1)/M^2$ in the heat affected zone to the reference value in the thermally undisturbed zone. As the thermal Mach number M approaches infinity, the limit value of the discontinuity across the shock surface is 4. (5) In transition of the thermal Mach number from the subsonic, transonic, to the supersonic ranges, the *r-dependency* of the near-tip temperature switches from $r^{1/2}$, r, to r^2. As the radial distance r measured from the crack tip

NOMENCLATURE

A, B	arbitrary constant in the eigenfunction $H(\theta)$
c	parameter used in the thermal wave equation, $v/2\alpha$ [$\mathrm{m^{-1}}$]
C	speed of heat propagation in the solid [$\mathrm{m\,s^{-1}}$]
C_p	heat capacity [$\mathrm{kJ\,kg^{-1}\,°C^{-1}}$]
k	thermal conductivity [$\mathrm{W\,m^{-1}\,°C^{-1}}$]
M	thermal Mach number, v/C
n	integer in the expression of eigenvalues
q, q_i	heat flux vector and its components in the ith direction [$\mathrm{W\,m^{-2}}$]
Q_i	angular variations of the near-tip heat flux components, $i = 1, 2$
r, θ	polar coordinates centered at the crack tip
R	instantaneous radius of curvature of the crack trajectory [m]
s	exponent of r for heat flux vector
S	general heat source term [$\mathrm{W\,m^{-3}}$]
t	physical time [s]
T	temperature [°C]
v	speed of the moving crack [$\mathrm{m\,s^{-1}}$].

Greek symbols

α	thermal diffusivity [$\mathrm{m^2\,s^{-1}}$]
γ	transformation function on the independent variable θ
Γ	time function in the near-tip temperature distribution
ζ	transformed variable from θ
η	dummy variable of integration
θ	polar angle of the moving coordinate system [deg]
λ	eigenvalues, exponent of r for the angular variation of near-tip temperature
ξ_i	moving coordinates centered at the crack tip, $i = 1, 2$ [m]
ρ	mass density [$\mathrm{kg\,m^{-3}}$]
Φ	transformation function on the dependent variable H
ω	angular velocity of the moving crack [$\mathrm{rad\,s^{-1}}$].

Other symbol

∇	gradient operator [$\mathrm{m^{-1}}$].

Subscripts and superscripts

$(\ )_{,i}$	$\partial/\partial\xi_i$, $i = 1, 2$
$(\ )^{\mathrm{D}}$	quantity in the thermal diffusion model.

approaches zero, the temperature gradient presents a $1/\sqrt{r}$-singularity only in the subsonic range with $M < 1$. These salient features are pertinent to the thermal wave model and cannot be depicted by the thermal diffusion model.

In the present study, the physical phenomena summarized above from (1) to (5) are to be further examined by considering the heat flux field around a crack tip propagating at various speeds in the subsonic, transonic, and supersonic ranges. Because the heat flux vector is related to the temperature by an integro-differential equation in the thermal wave model (refs. [1, 23–25] for instance), the flux formulation employed in this work is especially useful if a flux-specified condition is involved at the crack surfaces. This special feature of the thermal wave model has been clearly indicated by Frankel *et al.* [25].

In summary, the energy and constitutive equations in the thermal wave model are

$$-\nabla\cdot\mathbf{q}+S = \rho C_p T_{,t}$$

$$(\alpha/C^2)\mathbf{q}_{,t}+\mathbf{q} = -k\nabla T \qquad (1a)$$

where ρ is the density of the medium, k the thermal conductivity, C_p the heat capacity, S the body heat source, α the thermal diffusivity, and C the finite speed of heat propagation in the solid. Eliminating either temperature T or heat flux $\mathbf{q}$ from these equations, respectively, results in a field equation with flux ($\mathbf{q}$) or temperature (T) representations

$$\nabla[\nabla\cdot\mathbf{q}]-\nabla S = (1/\alpha)[(\alpha/C^2)\mathbf{q}_{,tt}+\mathbf{q}_{,t}] \qquad (1b)$$

$$\alpha\nabla^2 T+(1/\rho C_p)[S+(\alpha/C^2)S_{,t}] = (\alpha/C^2)T_{,tt}+T_{,t}. \qquad (1c)$$

An apparent heat source term containing the first-order derivatives of S with respect to space in the $\mathbf{q}$-representation and that with respect to time in the T-representation are obtained.

HEAT FLUX FORMULATION

The geometrical configuration of the moving crack under study is shown in Fig. 1. The crack is assumed

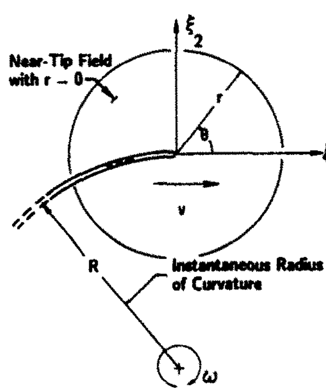

FIG. 1. Geometrical configuration of a moving crack and the material coordinate system convecting with the crack tip.

to be in the macroscopic level and propagating with constant linear and angular velocities (v and ω, respectively) along a smooth but arbitrary trajectory in an infinite medium with finite speed of heat propagation. The coordinates ξ_i with $i = 1, 2$ are the Lagrangian coordinate system convecting with the crack tip which are respectively the instantaneous tangent and normal to the crack surface.

Employing the thermal wave model, the use of **q**- or T-representations depends on the type of boundary conditions. This is a situation resulting from a complicated integro-differential relationship between the temperature and the heat flux vector in the hyperbolic wave model

$$q_i = -(C^2k/\alpha)\exp(-C^2t/\alpha) \times \int_0^t T_{,i}(\eta)\exp(C^2\eta/\alpha)\,d\eta. \quad (2)$$

In an earlier work [1], the T-representation has been adopted to investigate the near-tip behavior of temperature around a moving crack tip with *temperature* specified at the crack surface. For the same crack subject to a *flux-specified* surface condition, on the other hand, the **q**-representation of the energy equation is more convenient to use [25] which has been shown by equation (1b)

$$\alpha\nabla(\nabla\cdot\mathbf{q}) = (\alpha/C^2)\mathbf{q}_{,tt}+\mathbf{q}_{,t} \quad (3)$$

where the energy dissipating from the crack tip into the surrounding continua has been assumed to be negligible, namely $S(x_i, t) = 0$, during the formation of new crack surfaces. Under a description made by an observer moving with the crack tip, referring to Fig. 1, the Galileí transformation

$$\xi_1 = x_1+vt \quad \text{and} \quad \xi_2 = x_2 \quad (4)$$

can be applied and the time derivatives on the right-hand side of equation (3) are replaced by the material derivatives

$$\begin{aligned}
&\mathbf{q}_{,t} \to \mathbf{q}_{,t}-v\mathbf{q}_{,1}-\omega(\xi_1\mathbf{q}_{,2}-\xi_2\mathbf{q}_{,1})\\
&\mathbf{q}_{,tt} \to \mathbf{q}_{,tt}-2v\mathbf{q}_{,t1}+v^2\mathbf{q}_{,11}-\omega\xi_1[\mathbf{q}_{,t2}-(v-\omega\xi_2)\mathbf{q}_{,12}\\
&\quad+\omega\mathbf{q}_{,1}-\omega\xi_1\mathbf{q}_{,22}]+\omega\xi_2[\mathbf{q}_{,t1}-(v-\omega\xi_2)\mathbf{q}_{,11}-v\mathbf{q}_{,2}\\
&\quad-v\xi_1\mathbf{q}_{,12}]+\omega[\xi_2\mathbf{q}_{,t1}-\xi_1\mathbf{q}_{,t2}]\\
&\quad+v\omega[\xi_1\mathbf{q}_{,12}+\mathbf{q}_{,2}-\xi_2\mathbf{q}_{,11}]. \quad (5)
\end{aligned}$$

Subscripts i and t denote the differentiations with respect to the spatial coordinates ξ_i and the time variable t, respectively. In terms of the indicial notations, then, the combination of equations (3) and (5) yields

$$\begin{aligned}
\alpha q_{j,ji} &= \alpha M^2q_{i,11}+\{(\alpha/C^2)q_{i,tt}-(M^2/c)q_{i,1t}\}\\
&\quad+\{q_{i,t}-vq_{i,1}\}-\omega\{\xi_1q_{i,2}-\xi_2q_{i,1}\}\\
&\quad+[M^2/2cR]\{\xi_2[q_{i,t1}-(v-\omega\xi_2)q_{i,11}-\omega q_{i,2}\\
&\quad-\omega\xi_1q_{i,12}]-\xi_1[q_{i,t2}-(v-\omega\xi_2)q_{i,12}+\omega q_{i,1}\\
&\quad-\omega\xi_1q_{i,22}]-(\xi_1q_{i,t2}-\xi_2q_{i,t1})\\
&\quad+v(q_{i,2}+\xi_1q_{i,12}-\xi_2q_{i,11})\} \quad (6)
\end{aligned}$$

where a thermal Mach number M weighing the ratio of the crack propagation speed to the heat propagation speed, i.e. v/C, in the solid has been introduced and c a parameter defined as $v/2\alpha$. As the crack is stationary ($v = \omega = 0$) or the speed of heat propagation in the solid approaches infinity ($C \to \infty$), M approaches zero and equation (6) is reduced to that employing the thermal diffusion model.

By noticing that the ξ_2-axis is always normal to the propagating crack, the eigenstate of equation (6) for the heat flux components q_i are to be determined subject to the boundary conditions

$$q_2 = 0 \quad \text{at} \quad \xi_2 = 0 \quad \text{and} \quad \xi_1 < 0 \quad (7)$$

$$q_{2,2} = 0 \quad \text{at} \quad \xi_2 = 0 \quad \text{and} \quad \xi_1 > 0 \quad (8)$$

due to the symmetry of the problem. In the sequel we shall call the boundaries specified by equations (7) and (8) the *trailing* and the *leading* edges of the moving crack tip.

THE NEAR-TIP BEHAVIOR OF THE HEAT FLUX COMPONENT q_2

The field equation (6) is a partial differential equation with variable coefficients. A general form of the analytical solution is difficult to obtain. It should be noted, however, that the near-tip solution in the problem with a crack always gives the most important information. Such a solution does not only provide characteristics of the physical quantities varying in the neighborhood of the crack tip, but also an efficient algorithm of using singular tip elements in the finite element method [26]. In this section, therefore, we shall make an attempt to find a particular solution satisfying equations (6)–(8) as the crack tip is closely approached.

A product form of the solution for q_i is sought

$$q_i(r,\theta,t) = r^s\Gamma(t)Q_i(\theta), \quad i = 1,2 \quad (9)$$

with r and θ being the polar coordinates centered at the crack tip and $\Gamma(t)$ and $Q_i(\theta)$ the functions governing the time and the angular dependencies of the near-tip heat flux components. The coordinates (r, θ) are especially useful in describing the state of affairs in the vicinity of the crack tip. As usual, their gradients relate to those in the ξ_i coordinates according to

$$q_{i,1} = (\cos\theta)q_{i,r}-(\sin\theta/r)q_{i,\theta}$$

and

$$q_{i,2} = (\sin\theta)q_{i,r}+(\cos\theta/r)q_{i,\theta}, \quad \text{etc.} \quad (10)$$

Substituting equations (9) and (10) into equation (6), the *r-dependency* of the terms involved in the resulting equation can be summarized as follows:

$$\begin{aligned}
&q_{j,ji},\ q_{i,11} \sim r^{s-2};\\
&q_{i,1},\ q_{i,2},\ q_{i,t1},\ \xi_2q_{i,11},\ \xi_1q_{i,12},\ \xi_2q_{i,11} \sim r^{s-1};\\
&q_{i,t},\ \xi_1q_{i,1},\ \xi_2q_{i,2},\ \xi_1q_{i,2},\ \xi_2q_{i,1},\ q_{i,tt},\ \xi_2q_{i,t1},\ \xi_2^2q_{i,11},\\
&\quad \xi_1\xi_2q_{i,12},\ \xi_1q_{i,t2},\ \xi_2q_{i,t1},\ \text{and}\ \xi_1^2q_{i,22} \sim r^s. \quad (11)
\end{aligned}$$

Multiplying the entire equation by r^{2-s}, we can see that the terms proportional to r^{s-1} and r^s are proportional to r and r^2, respectively. As r approaches zero, i.e. in the immediate vicinity of the crack tip, these terms vanish and the asymptotic form of equation (6) reduces to

$$q_{j,ji} = M^2 q_{i,11}, \quad \text{with} \quad i,j = 1,2. \tag{12}$$

They are the dominant terms appearing originally in equation (6) and equation (12) is valid for any function of $\Gamma(t)$. The near-tip heat flux components are characterized by a single parameter M. The time-independence of this equation indicates that the near-tip characteristics of q_i are the same for a crack propagating in either the transient or steady state. Also, we notice that the angular velocity ω of the moving crack with smooth trajectory has a higher order effect on the thermal field around the crack tip. These features of the near-tip solution are similar to those found for the displacement field by Achenbach and Bažant [27] and the near-tip temperature of ref. [1].

Equation (12) can be expressed explicitly in terms of its components

$$(1-M^2)q_{1,11}+q_{2,12} = 0 \tag{13}$$

$$q_{1,12}+q_{2,22}-M^2 q_{2,11} = 0. \tag{14}$$

In a similar manner to that for obtaining the T- or the **q**-representation for the energy equation, a single equation governing q_2 can be obtained by eliminating q_1 from equations (13) and (14). Combining the differentiation of equation (13) with respect to ξ_2 and that of equation (14) with respect to ξ_1, we obtain

$$q_{2,122}+(1-M^2)q_{2,111} = 0. \tag{15}$$

A particular form for this equation is

$$q_{2,22}+(1-M^2)q_{2,11} = 0 \quad \text{or} \quad \nabla^2 q_2 = M^2 q_{2,11} \tag{16}$$

which is sufficient for our purpose as far as the eigenstate of the near-tip heat flux is concerned. One should notice that equation (16) will transit from a parabolic, elliptic, to a hyperbolic type as the thermal Mach number transits from the subsonic ($M < 1$), transonic ($M = 1$), to the supersonic ($M > 1$) ranges. Some intrinsic variations of the near-tip behavior of q_2 are thus expected in such a transition. Equation (16) has the same form as that governing the near-tip temperature around the crack tip [1]. The method of variable transformation proposed in the previous study can thus be extended to the present problem in the full range of the thermal Mach number.

Substituting equation (9) for q_2 into equation (16), and applying the chain rule given by equation (10), the angular distribution $Q_2(\theta)$ is found to be governed by the following equation:

$$(1-M^2\sin^2\theta)Q_{2,\theta\theta}-[M^2(1-s)\sin 2\theta]Q_{2,\theta} + s\{s+M^2[(2-s)\cos^2\theta-1]\}Q_2 = 0. \tag{17}$$

It is a second-order ordinary differential equation with variable coefficients which has to be solved subject to the boundary conditions

$$Q_{2,\theta} = 0 \text{ at } \theta = 0 \quad \text{and} \quad Q_2 = 0 \text{ at } \theta = \pi \tag{18}$$

as derived from equations (7) and (8). In obtaining an analytical solution for $Q_2(\theta)$ satisfying equation (17), the method proposed previously involves a transformation on the dependent variable from Q_2 to Φ

$$Q_2(\theta) = (1-M^2\sin^2\theta)^{s/2}\Phi(\theta) \tag{19}$$

and another transformation on the independent variable from θ to γ governed by

$$(1-M^2\sin^2\theta)\gamma_{,\theta\theta}-(M^2\sin 2\theta)\gamma_{,\theta} = 0. \tag{20}$$

The function $\gamma(\theta)$ and the resulting form of equation (17) after these transformations depend on the thermal Mach number of the moving crack.

(a) *Subsonic range with* $M < 1$. In the case of $M < 1$, the function $\gamma(\theta)$ satisfying equation (20) can be integrated to give

$$\gamma(\theta) = \tan^{-1}[(1-M^2)^{1/2}\tan\theta]$$

and consequently

$$\gamma_{,\theta} = (1-M^2)^{1/2}/(1-M^2\sin^2\theta). \tag{21}$$

This is the transformation observed by Achenbach and Bažant [27] for the out-of-plane displacement field around a moving crack tip. Similar to the previous problem involving the near-tip temperature [1], it is derived analytically from equation (20) for the present problem. The resulting Φ function in the γ space takes a simple form in this case

$$\Phi_{,\gamma\gamma}+s^2\Phi = 0. \tag{22}$$

Subject to boundary condition (18) represented in terms of functions Φ and γ

$$\Phi_{,\gamma} = 0 \text{ at } \gamma = 0 \quad \text{and} \quad \Phi = 0 \text{ at } \gamma = \pm\pi \tag{23}$$

the solution for Φ can be obtained immediately as

$$\Phi = A\cos s\gamma \tag{24}$$

with A being the arbitrary constant in an eigenvalue system and the eigenvalue s satisfying the following equation:

$$s = (2n+1)/2, \quad \text{with integer } n = 0,1,2,\ldots \tag{25}$$

The fundamental mode possessing the lowest eigenvalue, namely $s = 1/2$ corresponding to $n = 0$, dominates the near-tip behavior of q_2. As r approaches zero in the vicinity of the crack tip, we notice that the terms of r^s, with s being shown in equation (25) but $n \neq 0$, will approach zero faster than that of $r^{1/2}$. Substituting equation (24) into equation (19) and taking the inverse transformation from γ to θ according to equations (21), the final form of the solution for $Q_2(\theta)$ is

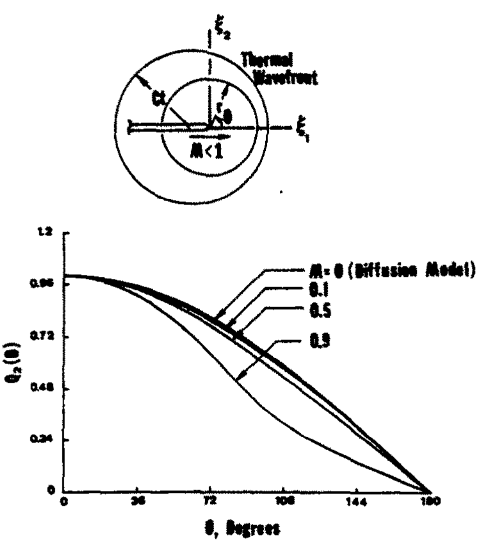

FIG. 2. Angular variations of the near-tip heat flux q_2 at thermal Mach numbers $M = 0$, 0.1, 0.5, and 0.9—subsonic range with $M < 1$.

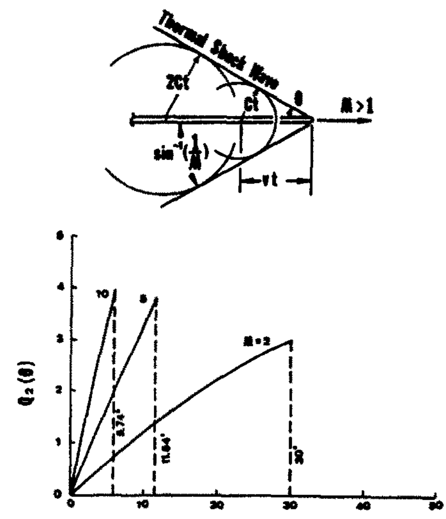

FIG. 3. Angular variations of the near-tip heat flux q_2 at thermal Mach numbers $M = 1$, 5, and 10—supersonic range with $M > 1$.

$$Q_2(\theta) = A[(1 - M^2 \sin^2 \theta)^{1/2} + \cos\theta]^{1/2}/\sqrt{2}, \quad \text{for} \quad M = 1 \tag{26}$$

which governs the angular distribution of q_2 in the core region. The corresponding expression in the thermal diffusion model can be retrieved by substituting M by zero in equation (26)

$$Q_2^D(\theta) = A[1 + \cos\theta]^{1/2}/\sqrt{2}. \tag{27}$$

This expression is valid for either a stationary crack ($v = 0$) or a medium with the heat propagation speed being infinity ($C \to \infty$). In the present case with $M < 1$, it should be noted, referring to equation (9), that the gradient $\partial q_2/\partial r$ in both the thermal wave and the thermal diffusion models displays a near-tip behavior of $1/\sqrt{r}$.

The graphical representations for $Q_2(\theta)$ and $Q_2^D(\theta)$ with A being unity are shown in Fig. 2. The angle θ is measured from the *leading edge* of the moving crack. As expected, the deviation between the two models becomes significant as the thermal Mach number increases. For all the cases with $M = 0$ (the diffusion model), 0.1, 0.5, and 0.9, the temperature reaches its maximum at the leading edge of the moving crack at $\theta = 0°$. Also, the Q_2-function (and hence the magnitude of the heat flux component q_2) decreases as the thermal Mach number increases.

(b) *Supersonic range with* $M > 1$. For the same crack moving to the right at a speed faster than that of the heat propagation in the solid, equation (19) can still be used but the distribution of $Q_2(\theta)$ under investigation must be confined to the domain of

$$0 \leqslant \theta \leqslant \sin^{-1}(1/M) \quad \text{or} \quad 0 \leqslant \theta \leqslant \tan^{-1}[1/(M^2 - 1)^{1/2}] \tag{28}$$

with θ being measured in the present case from the *trailing edge* of the moving crack, as shown in Fig. 3. For $M > 1$, alternatively, the transformation $\gamma(\theta)$ satisfying equation (20) takes the form

$$\gamma(\theta) = \tfrac{1}{2}\ln\{[(M^2 - 1)^{1/2}\tan\theta + 1]/[(M^2 - 1)^{1/2}\tan\theta - 1]\}$$

and

$$\gamma_{,\theta} = (M^2 - 1)^{1/2}/(1 - M^2 \sin^2\theta). \tag{29}$$

The corresponding equation to equation (22) for the Φ-function can be found as

$$\Phi_{,\gamma\gamma} - s^2\Phi = 0, \quad \text{for} \quad M > 1. \tag{30}$$

Obviously, the eigenvalue system formulated previously for $M < 1$ has a dramatic change for the present case. We first notice that the physical domain represented by equation (28) for $M > 1$ is identical to that for the heat affected zone induced by a moving crack subject to a *temperature-specified* condition at the crack surface [1]. Therefore, it is informative to conclude that the distribution of q_2 in the *heat affected zone* is governed by equations (19), (29), and (30). While the heat flux components stay at the initial value (assumed to be zero without loss in generality) in the rest of the physical domain which is defined as the *thermally undisturbed zone*. The heat affected zone and the thermally undisturbed zone are separated by a thermal shock wave located at $\theta = \sin^{-1}(1/M)$ measured from the *trailing edge* of the moving crack. This is another identical result for a moving crack subject to either a temperature- or a flux-specified condition at the crack surface.

Defining now a parameter ζ

$$\zeta(\theta) = [1 + (M^2 - 1)^{1/2}\tan\theta]/[1 - (M^2 - 1)^{1/2}\tan\theta] \tag{31}$$

such that from equation (29)

$$\gamma = \pm \frac{i\pi}{2} + \tfrac{1}{2}\ln(\zeta) \tag{32}$$

with $i = \sqrt{-1}$ and ζ being greater than zero for θ in the domain of the heat affected zone specified by equation (28). The solution of equation (30) in terms of the variable ζ can then be obtained as

$$\begin{aligned}\Phi(\zeta(\theta)) = {} & \cos(s\pi/2)\{A\exp[s\ln(\zeta)/2] \\ & + B\exp[-s\ln(\zeta)/2]\} \\ & + i\sin(s\pi/2)\{A\exp[s\ln(\zeta)/2] \\ & - B\exp[-s\ln(\zeta)/2]\}.\end{aligned} \tag{33}$$

Based on the argument that the heat flux vector must be real, equation (33) gives

$$\sin(s\pi/2) = 0, \quad \text{or} \quad s = 2n \quad \text{with} \quad n = 1, 2, 3, \ldots \tag{34}$$

For a non-trivial solution, therefore, the lowest eigenvalue s in the supersonic case is 2. This result intrinsically varies the near-tip behavior of the heat flux component q_2. In the subsonic range with $M < 1$, the r-dependency of q_2 is $\sqrt{r}$. While in the supersonic range with $M > 1$, such a dependency transits to r^2 which cannot be depicted by the thermal diffusion model. Notice also that only one boundary condition at $\theta = 0$ (the trailing edge of the moving crack), or $\zeta = 1$ and $\gamma = \pm i(\pi/2)$ according to equation (32), remains in the present case with $M > 1$

$$\Phi = 0 \quad \text{at} \quad \gamma = \pm i(\pi/2) \quad \text{and} \quad \zeta = 1. \tag{35}$$

Another boundary at $\theta = \pi$ (the leading edge of the moving crack) stays in the thermally undisturbed zone and the symmetrical condition $\Phi_{,\gamma} = \Phi_{,\theta} = 0$ is automatically satisfied for a uniform distribution of q_2 being zero (the reference value). Substituting equation (33) into equation (35) renders a result of $A = -B$. The Φ-function in the supersonic range is thus obtained as

$$\Phi(\zeta(\theta)) = B\left(\zeta - \frac{1}{\zeta}\right) \tag{36}$$

with B being an arbitrary constant. Combining equation (19) for $s = 2$ with equation (36) then gives the angular distribution $Q_2(\theta)$ in the heat affected zone

$$Q_2(\theta) = B(1 - M^2\sin^2\theta)\left(\zeta - \frac{1}{\zeta}\right) \tag{37}$$

which, upon substitution of equation (31), can be expressed in terms of the variable θ

$$Q_2(\theta) = 2B(M^2-1)^{1/2}\sin(2\theta), \quad \text{for} \quad M > 1. \tag{38}$$

Note that as the thermal shock wave is approached from the site in the heat affected zone, namely $\theta \to \sin^{-1}(1/M)$, the quantity of $\sin(2\theta)$ approaches $2(M^2-1)^{1/2}/M^2$ and we have

$$Q_2(\theta) \to 4B(M^2-1)/M^2, \quad \text{as} \quad \theta \to \sin^{-1}(1/M). \tag{39}$$

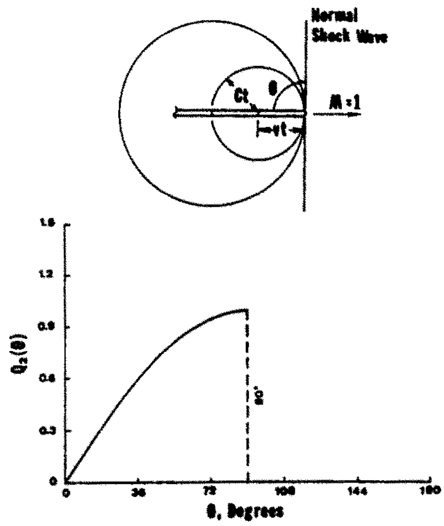

FIG. 4. Angular variations of the near-tip heat flux q_2 at the transonic stage with $M = 1$.

Equation (39) shows that the heat flux component q_2 remains bounded at the thermal shock wave located at $\theta = \sin^{-1}(1/M)$ with $M > 1$. It has a limit value of $4B$ as M approaches infinity. We also notice that the heat flux component q_2 is discontinuous but finite across the surface of the thermal shock wave. As the thermal Mach number increases, the Q_2-function represented by equation (38), and hence the heat flux component q_2 according to equation (9), increases. Together with the result obtained previously in the subsonic range that the magnitude of q_2 decreases as the thermal Mach number increases, this is a swinging phenomenon similar to that being found for the temperature variation in transition of the thermal Mach number from the subsonic to the supersonic ranges [1]. These phenomena are shown in Fig. 3 with M being 2, 5, and 10 in the supersonic range.

(c) *Transonic stage with $M = 1$.* As the speed of the moving crack is the same as that of the heat propagation in the solid, the transformation $\gamma(\theta)$ degenerates into the form of

$$\gamma(\theta) = \tan\theta. \tag{40}$$

The equation governing the Φ-function in this case becomes

$$\Phi_{,\gamma\gamma} = 0 \tag{41}$$

which possesses a solution of the form

$$\Phi(\gamma) = A\gamma + B. \tag{42}$$

The angle θ, similar to the supersonic case with $M > 1$, is measured from the trailing edge of the moving crack as shown in Fig. 4. The transonic case is the one for the onset of thermal shock formation. According to equation (28), a normal shock starts to form at $\theta = 90°$ in the present case with $M = 1$. The heat affected zone consequently ranges from 0° to 90° measured from the crack surface. Only the boundary condition at $\theta = 0°$, or $\gamma = 0$ according to equation (40), needs to be considered

$$\Phi = 0 \quad \text{at} \quad \gamma = 0. \tag{43}$$

From equation (42), we have $B = 0$ and A an arbitrary constant. Determining the values of s at the transonic stage relies on the argument made on the *boundedness* of the heat flux across the shock surface. This is a general feature obtained in the supersonic range (referring to equation (39)) and should be satisfied as the thermal Mach number approaches 1. Substituting equation (40) into equation (42) and the result into equation (19), the Q_2-function is thus obtained

$$Q_2(\theta) = A\cos^s\theta\tan\theta, \quad \text{for} \quad M = 1. \tag{44}$$

Obviously, the lowest value of s must be equal to 1 such that

$$Q_2(\theta) = A\sin\theta, \quad \text{for} \quad M = 1 \tag{45}$$

and the angular distribution of $Q_2(\theta)$ remains bounded as θ approaches 90°. Again, we notice that the r-dependency of q_2 represented by equation (9) switches from $r^{1/2}$ to r as the thermal Mach number transits from the subsonic range to the transonic stage. It further switches to r^2 as the thermal Mach number transits into the supersonic range. The graphical representation of $Q_2(\theta)$ at $M = 1$ is shown in Fig. 4.

For the supersonic (case b) and the transonic (case c) cases with $M \geqslant 1$, as shown in Figs. 3 and 4, the magnitude of $Q_2(\theta)$ in the heat affected zone decays from the thermal shock wave to the trailing edge of the moving crack.

DISCUSSIONS

The angular distribution of the heat flux component q_1

The function of $Q_1(\theta)$, referring to equation (9), governing the angular distribution of the heat flux component q_1 around the crack tip can be found according to the corresponding expressions of $Q_2(\theta)$ in the respective ranges of the thermal Mach number. With the assistance of equation (10), substituting equation (9) for $i = 1$ and 2 into equations (13) and (14) renders a set of coupled equations for Q_1 and Q_2 in the near-tip field. Eliminating the terms containing $Q_{2,\theta}$ in these equations then gives a first-order ordinary differential equation governing $Q_1(\theta)$

$$(1 - M^2\sin^2\theta)Q_{1,\theta} + \left(\frac{sM^2}{2}\sin 2\theta\right)Q_1 = sQ_2 \tag{46}$$

where the function $Q_2(\theta)$ is the corresponding expression in the subsonic (equation (26)), transonic (equation (45)), and supersonic (equation (38)) ranges. This equation has an integrating factor $(1 - M^2\sin^2\theta)^{-s/2}$ and its solution can be obtained immediately as

$$Q_1(\theta) = s(1 - M^2\sin^2\theta)^{s/2} \times \int^{\theta} Q_2(\eta)/(1 - M^2\sin^2\eta)^{[(s+2)/2]}\,d\eta. \tag{47}$$

The r-dependency of $\partial\mathbf{q}/\partial r$

On employing Fourier's law of heat conduction, the heat flux vector has the same r-dependency as that of the *temperature gradient* $\partial T/\partial r$, as shown by Sih [28] for example. On employing the thermal wave model, on the other hand, the r-dependency of the heat flux vector is the same as that of the temperature *itself*. This can be seen by comparing the present results with those obtained previously for the near-tip temperature [1]. The r-dependency of $\partial\mathbf{q}/\partial r$ depends on the thermal Mach number of the moving crack. It switches from $1/\sqrt{r}$ to r as the thermal Mach number transits from the subsonic to the supersonic ranges. At the transonic stage with $M = 1$, $\partial\mathbf{q}/\partial r$ is independent of r. As the radial distance r approaches zero, the quantity $\partial\mathbf{q}/\partial r$ presents a $1/\sqrt{r}$-singularity only in the subsonic range with $M < 1$ while it is bounded at the transonic and in the supersonic ranges with $M \geqslant 1$. This result shows that as the speed of the moving crack exceeds the heat propagation speed in the solid, the thermal energy accumulated at the crack tip is not as pronounced.

CONCLUSIONS

Due to the similarity of the mathematical structures between the temperature (T) and the heat flux ($\mathbf{q}$) formulations, the physical phenomena of thermal shock formation are identical for a moving crack subject to either a temperature- or a flux-specified condition at the crack surface. The thermal shock wave starts to form at the transonic stage and sweeps toward the trailing edge of the moving crack as the thermal Mach number further increases to the supersonic range. The thermal shock angle is obtained as $\sin^{-1}(1/M)$ for $M \geqslant 1$ and the physical domain of the heat affected zone is found to be $0 \leqslant \theta \leqslant \sin^{-1}(1/M)$ with the angle θ being measured from the crack surface. The heat flux components in the heat affected zone are represented by the combination of equations (9), (38), (45), and (47) while they stay at a reference value in the thermally undisturbed zone. Similar to the temperature in the near field of the crack tip, the near-tip heat flux also has a discontinuous but finite change across the surfaces of the thermal shock wave. This is clearly shown by equation (39).

Generally speaking, the physical phenomena of thermal shock formation induced by a *moving crack* are quite similar to those induced by a *moving heat source* [23, 24]. The difference, however, lies in that both the temperature and the heat flux vector approach *infinity* as the thermal shock wave induced by a moving heat source is approached from the site in the heat affected zone, while these physical quantities keep *finite* thereby if the thermal shock wave is induced by a moving crack. In transition of the ther-

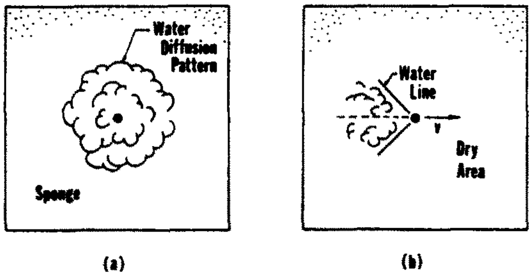

FIG. 5. A chunk of sponge impinged by (a) a stationary water jet and (b) a moving water jet with a velocity v.

mal Mach number from the subsonic to the supersonic ranges, the variation of the r-dependency of the heat flux vector is another salient feature described by the thermal wave model. Such a behavior in transition is also the same as that for the near-tip temperature. Together with the swinging phenomenon discussed above, these important physical phenomena are pertinent to the thermal wave model.

Although the thermal shock formation is a direct consequence of simulating the thermal disturbance propagating at a finite speed in the solid, an important motivation of the present work is to attract the interest from researchers such that direct evidence for the existence of thermal shock waves can be established. In the absence of experimental support for the thermal shock formation, a qualitative description could be made by considering a water jet impinging on the surface of a chunk of sponge as illustrated in Fig. 5. When the water jet is held stationary, as shown in Fig. 5(a), a distinct water diffusion pattern emanating from the point of impingement can be observed. When the water jet is moving with a velocity v as shown in Fig. 5(b), on the other hand, two distinct 'water lines' will be observed instead and they separate the dry area (moisture undisturbed zone) from the region with water concentration (moisture affected zone). Because the wave speed of mass transfer in the sponge is comparatively slow, such a phenomenon is easier to explain according to commonsense. In the problem involving the heat transfer in the solid, because the wave speed of heat propagation is much faster than that of the mass diffusion in the sponge, the experiments become more involved and some advanced techniques may be necessary. For the present problem involving a propagating crack tip, the example given above should also be illustrative because we have shown that the formation of the thermal shock waves is similar to that induced by a moving heat source.

An immediate topic followed is to understand the resulting thermoelastic stress field around the crack tip. Under regular conditions, a well-known fact in Linear Elastic Fracture Mechanics is that the near-tip stress field has a $1/\sqrt{r}$-singularity as the crack tip is closely approached. Under the interactions between the thermal and the mechanical fields, how the transition of the r-dependency of the thermal field influence the stress singularity is certainly worthy of study. We will leave this interesting topic for future communications.

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ONDES DE CHOC THERMIQUE INDUITES PAR UNE FISSURE MOBILE—UNE FORMULATION DE FLUX THERMIQUE

Résumé—Le phénomène de choc thermique induit par la propagation rapide d'une fissure dans un solide a éte récemment étudié à travers une formulation en température (*ASME J. Heat Transfer* **112**, 21–27 (1990)). De façon à confirmer cette unique approche des ondes thermiques émanant de la lèvre de la fissure, l'étude présente une formulation de flux et examine le même phénomène physique d'un autre point de vue. Les discontinuités du choc thermique résulte de l'accumulation d'énergie dans une direction préférentielle lorsque la vitesse de propagation de la fissure dépasse la vitesse de propagation dans le solide. En employant la formulation du flux on étudie la formation du choc thermique, les évolutions de la zone affectée par la chaleur et la zone thermiquement non perturbée et la transition de la dépendence vis-à-vis de r du vecteur flux de chaleur dans la transition du nombre de Mach thermique depuis le régime subsonique jusqu'au régime supersonique. L'analogie entre la température et le flux thermique dans la région proche de la lèvre est établie et le phénomène de choc thermique est confirmé d'un point de vue théorique.

THERMISCHE SCHOCKWELLEN AN EINER RISSSPITZE—BESCHREIBUNG DES WÄRMESTROMS

Zusammenfassung—Mit Hilfe einer thermischen Betrachtung wurde kürzlich das Phänomen des thermischen Schocks untersucht, der durch einen schnell durch einen Festkörper laufenden Riß verursacht wird (*ASME J. Heat Transfer* **112**, 21–27 (1990)). Die vorliegende Arbeit benutzt die Beschreibung des Wärmestroms, um die besonderen Merkmale weiter zu bestätigen, die aus früheren Forschungsarbeiten für thermische Wellen, die von Rissen herrühren, erhalten wurden. Das gleiche physikalische Phänomen wird nun von einem anderen Standpunkt aus betrachtet. Die thermischen Schockwellen ergeben sich aus der Anhäufung von Energie in einer bevorzugten Richtung, wenn die Geschwindigkeit des fortschreitenden Risses die Ausbreitungsgeschwindigkeit des Wärmestroms im Festkörper überschreitet. Durch Beschreibung des Wärmestroms im Modell für die thermischen Wellen wird in der vorliegenden Arbeit die Entstehung des thermischen Schocks untersucht, außerdem das Verhalten der thermisch gestörten und der ungestörten Zone, sowie der Übergang der r-Abhängigkeit des Wärmestromvektors beim Übergang der thermischen Mach-Zahl vom Unterschall- in den Überschall-Bereich. Die Analogie zwischen Temperatur und Wärmestrom im Bereich der Rißspitze wird formuliert, und die Phänomene des thermischen Schocks werden von einem theoretischen Standpunkt aus bestätigt.

ТЕПЛОВЫЕ УДАРНЫЕ ВОЛНЫ, ИНДУЦИРОВАННЫЕ РАСПРОСТРАНЯЮЩЕЙСЯ ТРЕЩИНОЙ—ФОРМУЛИРОВКА ТЕПЛОВОГО ПОТОКА

Аннотация—В недавно опубликованной работе (Труды АСМЕ, Теплопередача **112**, 21–27 (1990)) явления термического удара, индуцированные быстро распространяющейся в твердом теле вершиной трещины, описывались с помощью температурных величин. Для дальнейшего подтверждения установленных в этом исследовании уникальных свойств тепловых волн, испускаемых из вершины трещины, в настоящей работе используется потоковая формулировка, а затем те же физические явления исследуются под другим углом зрения. Разрыв непрерывности в виде термического удара возникает из-за накопления энергии в определенном направлении, когда скорость распространения вершины трещины становится больше скорости распространения тепла в твердом теле. Используя потоковую формулировку в модели тепловой волны, исследуются развитие термического удара, эволюция термически возмущенных и инертных зон и переход зависимости вектора теплового потока от направления r в переход теплового числа Маха из дозвукового в сверхзвуковой диапазон. Установлена аналогия между температурным и тепловым потоками у вершины трещины, и дано теоретическое подтверждение явлениям термического удара.